LECTURE NO 10

If $\mathbf{G}(r) = 10e^{-2z}(\rho \mathbf{a}_\rho + \mathbf{a}_z)$, determine the flux of $\mathbf{G}$ out of the entire surface of the cylinder $\rho = 1, 0 \leq z \leq 1$. Confirm the result using the divergence theorem.

If Ψ is the flux of **G** through the given surface, shown in Figure 3.17, then

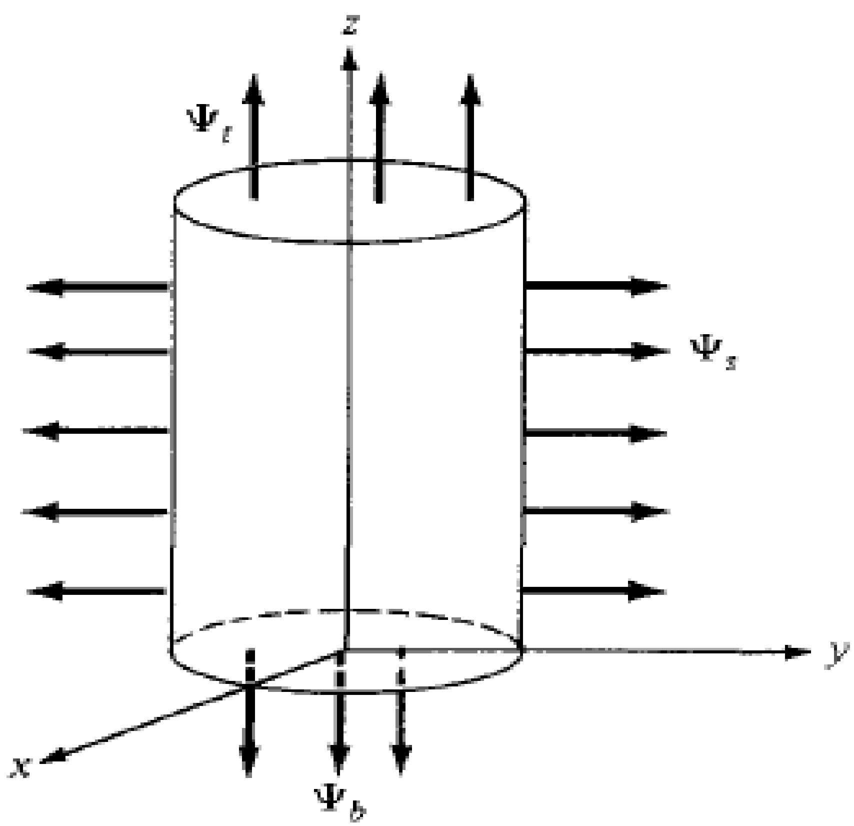
$$\Psi = \oint \mathbf{G} \cdot d\mathbf{S} = \Psi_t + \Psi_b + \Psi_s$$

where Ψ_t, Ψ_b, and Ψ_s are the fluxes through the top, bottom, and sides (curved surface) of the cylinder as in Figure 3.17.

For Ψ_t, $z = 1$, $d\mathbf{S} = \rho\, d\rho\, d\phi\, \mathbf{a}_z$. Hence,

$$\Psi_t = \int \mathbf{G} \cdot d\mathbf{S} = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} 10e^{-2} \rho\, d\rho\, d\phi = 10e^{-2}(2\pi) \left.\frac{\rho^2}{2}\right|_0^1$$

$$= 10\pi e^{-2}$$



z
Ψ_t
Ψ_s
y
x
Ψ_b

For Ψ_b, $z = 0$ and $d\mathbf{S} = \rho\, d\rho\, d\phi(-\mathbf{a}_z)$. Hence,

$$\Psi_b = \int_b \mathbf{G} \cdot d\mathbf{S} = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} 10e^{0} \rho\, d\rho\, d\phi = -10(2\pi) \frac{\rho^2}{2} \Big|_0^1$$
$$= -10\pi$$

For Ψ_s, $\rho = 1$, $d\mathbf{S} = \rho\, dz\, d\phi\, \mathbf{a}_\rho$. Hence,

$$\Psi_s = \int_s \mathbf{G} \cdot d\mathbf{S} = \int_{z=0}^{1} \int_{\phi=0}^{2\pi} 10e^{-2z} \rho^2\, dz\, d\phi = 10(1)^2(2\pi) \frac{e^{-2z}}{-2} \Big|_0^1$$
$$= 10\pi(1 - e^{-2})$$

Thus,

$$\Psi = \Psi_t + \Psi_b + \Psi_s = 10\pi e^{-2} - 10\pi + 10\pi(1 - e^{-2}) = 0$$

Alternatively, since S is a closed surface, we can apply the divergence theorem:

$$\Psi = \oint_S G \cdot d\mathbf{S} = \int_v (\nabla \cdot \mathbf{G})\, dv$$

But

$$\nabla \cdot \mathbf{G} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho G_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi} G_\phi + \frac{\partial}{\partial z} G_z$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2 10 e^{-2z}) - 20 e^{-2z} = 0$$

showing that **G** has no source. Hence,

$$\Psi = \int_v (\nabla \cdot G)\, dv = 0$$